THE APPROXIMATION OF BOUNDARY CONDITIONS OF THE THIRD KIND IN CERTAIN PROBLEMS OF STEADY-STATE HEAT CONDUCTION

E. A. Sidorov

UDC 536.21

We examine the error which results from the approximation of boundary conditions of the third kind by boundary conditions of the first kind, in the analytical solution of the Laplace equation for nonuniform regions and mixed boundary conditions.

In the theoretical and experimental solution of a number of problems relating to steady-state heat transfer, particularly extensive attention was devoted to the method based on the substitution of the thermal resistance of the heat boundary layer between the solid and the ambient medium by the thermal resistance – equal in magnitude – of a conditional additional layer of the solid material under consideration.

In theoretical investigations the area of application for this method is generally limited to complex one-dimensional problems. As regards two-dimensional problems, the problem of the applicability to these of the method of the additional layer is as yet, to the best of our knowledge, totally untreated in the literature.

The analysis presented below is devoted to the problem of a penetrating and heat-conducting inclusion.

1. Formulation of the Problem. We are to determine the temperature field $\tau(x', y')$ in a plate of thickness δ, provided that the temperatures of the medium are specified (t_{in}) as well as the heat-transfer coefficients (α_{in} and α_{out}) for the inside ($y' = 0$) and outside ($y' = \delta$) surfaces, respectively. The plate contains a penetrating rectangular inclusion of width a. The coefficients of thermal conductivity for the inclusion and the plate are, respectively, equal to λ_1 and λ_2.

In dimensionless quantities the problem reduces to the solution of the differential equation

$$\frac{\partial^2\theta_i}{\partial x^2} + \frac{\partial^2\theta_i}{\partial y^2} = 0 \tag{1}$$

with the following boundary conditions:

$$\frac{\partial\theta_1(0, y)}{\partial x} = 0, \tag{2}$$

$$\frac{\partial\theta_2(\infty, y)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial\theta_i(x, 0)}{\partial y} - \mathrm{Bi}\,\theta_i(x, 0) = 0, \tag{4}$$

$$\frac{\partial\theta_i(x, 1)}{\partial y} + k\,\mathrm{Bi}\,\theta_i(x, 1) = k\,\mathrm{Bi}, \tag{5}$$

$$\theta_1(\xi, y) = \theta_2(\xi, y), \tag{6}$$

$$\lambda_1\frac{\partial\theta_1(\xi, y)}{\partial x} = \lambda_2\frac{\partial\theta_2(\xi, y)}{\partial x}. \tag{7}$$

Scientific-Research Institute for Standard and Experimental Planning, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 1, pp. 135-139, July, 1969. Original article submitted June 18, 1968.

2. The Exact Solution. Let us present the function θ_i in the form of the sum of two functions

$$\theta_i(x, y) = u_i(y) + v_i(x, y), \tag{8}$$

the first of which satisfies the ordinary differential equation

$$\frac{d^2u_i}{dy^2} = 0 \tag{9}$$

and the original nonuniform boundary conditions (4) and (5), while the second function satisfies the original Laplace equation (1), (2), (3), (6), and (7), as well as the uniform boundary conditions derived from (4) and (5) by replacing the right-hand member in (5) by zero.

Having solved (9) in conjunction with (4) and (5), we find that

$$u_i = \frac{1 + \mathrm{Bi}\, y}{1 + \frac{1}{k} + \mathrm{Bi}} \tag{10}$$

and, consequently,

$$u_{\mathrm{Bi}} = \frac{1}{1 + \frac{1}{k} + \mathrm{Bi}}. \tag{11}$$

To find the function $v_i(x, y)$ we will use the Fourier method of separating variables.

Assuming that

$$v_i(x, y) = X_i(x)\, Y_i(y), \tag{12}$$

we find from (1) two ordinary differential equations whose solution for the corresponding boundary conditions will be

$$X_1 = C_1 \operatorname{ch} \beta_1 x, \tag{13}$$

$$X_2 = C_2 \exp(-\beta_2 x), \tag{14}$$

$$Y_i = \beta_i \cos \beta_i y + \mathrm{Bi} \sin \beta_i y. \tag{15}$$

Consequently, using the Fourier series expansion of the eigenfunctions, we have

$$v_i = \sum_{n=1}^{\infty} C_{in} X_{in} Y_{in}, \tag{16}$$

where the summation is performed over successively increasing positive roots of the characteristic equation

$$\operatorname{ctg} \beta_i = \frac{\beta_i^2 - k\,\mathrm{Bi}^2}{\beta_i (k+1)\,\mathrm{Bi}}. \tag{17}$$

Finally, having determined from (6) and (7) the values of the constants C_{1n} and C_{2n}, we find the unknown function for the temperature field in the zone of the inclusion (θ_1) and in the zone of the plate itself (θ_2):

$$\theta_1 = u_1 - \sum_{n=1}^{\infty} \frac{\Psi_{1n} Y_{1n} \operatorname{ch} \beta_{1n} x}{\operatorname{ch} \beta_{1n} \xi + \frac{\lambda_1 \beta_{1n}}{\lambda_2 \beta_{2n}} \operatorname{sh} \beta_{1n} \xi}, \tag{18}$$

$$\theta_2 = u_2 + \sum_{n=1}^{\infty} \frac{\Psi_{2n} Y_{2n} \exp[-\beta_{2n}(x - \xi)]}{1 + \frac{\lambda_2 \beta_{2n}}{\lambda_1 \beta_{1n}} \operatorname{cth} \beta_{1n} \xi}, \tag{19}$$

where

$$\Psi_{in} = \frac{\int_0^1 (u_1 - u_2) Y_{in}\, dy}{\int_0^1 Y_{in}^2\, dy}. \tag{20}$$

TABLE 1. Relative Temperature $\Delta\theta_{in}$ at the Boundary Between Two Semiinfinite Plates (Bi = 2.25, k = 8/3)

λ_1/λ_2	1	1,67	2,0	2,5	3,33	5	10	∞
According to [1]	0	0,044	0,062	0,085	0,112	0,141	0,176	0,215*
According to (30)	0	0,050	0,070	0,095	0,126	0,164	0,213	0,276

*Extrapoluted

3. Approximate Solution. Within the framework of the above-cited approximation we will conditionally increase the plate thickness δ by λ_i/α_{in} from the inside and by λ_i/α_{out} from the outside. In dimensionless coordinates this will indicate a conditional thickening by 1/Bi and by 1/kBi, respectively.

As before, we will seek the solution in the form of sum (8) of two functions. The first of these functions must satisfy (9) and the nonuniform binary conditions:

$$u_i\left(-\frac{1}{\mathrm{Bi}}\right)=0, \quad (21)$$

$$u_i\left(1+\frac{1}{k\,\mathrm{Bi}}\right)=1. \quad (22)$$

As regards the second function v_i, it must satisfy the uniform boundary conditions with respect to the y coordinate:

$$v_i\left(x,\ -\frac{1}{\mathrm{Bi}}\right)=0, \quad (23)$$

$$v_i\left(x,\ 1+\frac{1}{k\,\mathrm{Bi}}\right)=0, \quad (24)$$

and conditions (2) and (3) with respect to the x coordinate.

Using the Fourier method and performing transformations similar to those in section 2, we finally obtain

$$\theta_1=u_1-\sum_{n=1}^{\infty}\frac{\Psi_{1n}\,\mathrm{ch}\,n\pi x\sin n\pi u_1}{\mathrm{ch}\,n\pi\xi+\frac{\lambda_1}{\lambda_2}\,\mathrm{sh}\,n\pi\xi}, \quad (25)$$

$$\theta_2=u_2+\sum_{n=1}^{\infty}\frac{\Psi_{2n}\exp\left[-n\pi(x-\xi)\right]\sin n\pi u_2}{1+\frac{\lambda_2}{\lambda_1}\,\mathrm{cth}\,n\pi\xi}, \quad (26)$$

where

$$\Psi_{in}=\frac{2\left(u_{\mathrm{in}1}-u_{\mathrm{in}2}\right)\left[1+\frac{1}{k}(-1)^n\right]}{u_{\mathrm{Bi}}\,\mathrm{Bi}\,n\pi}, \quad (27)$$

and the values of u_i and u_{Bi}, as before, are determined from (10) and (11).

The advantage of the approximate solution (25) and (26) in comparison with the exact solution (18) and (19) lies in the comparative simplicity of the norming factor (27) and most importantly in the fact that there is no need to solve the two-parameter transcendental equation (17).

4. Evaluation of Approximation Accuracy. From (25) and (26), assuming that $x=\xi$ and $y=0$, we can determine the dimensionless temperature of the inside surface at the boundary of the inclusion zone:

$$\theta_{\mathrm{in}}(\xi)=u_{\mathrm{in}2}+\sum_{n=1}^{\infty}\frac{\Psi_{2n}\sin n\pi u_{\mathrm{B}2}}{1+\frac{\lambda_2}{\lambda_1}\,\mathrm{cth}\,n\pi\xi}. \quad (28)$$

If the width a of the inclusion is sufficiently great, we can assume that $\xi \to \infty$. When two semiinfinite plates are in contact, formula (28) admits of substantial simplifications. Indeed, since

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi z = \frac{\pi}{2}(1 - z),$$

for $\xi \to \infty$ from (28) we find

$$\Delta\theta_{in} = \theta_{in}(\xi) - u_{in2} = \frac{u_{in1} - u_{in2}}{1 + \frac{\lambda_2}{\lambda_1}}.$$

Table 1 gives a comparison of the values of $\Delta\theta_{in}$ found by the method of electrical modeling [1] and is in rather good agreement with the results of the exact solution, with the values calculated from (30).

As we can see from the data in the table, in the range $\lambda_1/\lambda_2 > 1.5$ of practical utilization the approximate solution yields values for $\Delta\theta_{in}$ that are overstated by approximately 15%. In this case, with an increase in the parameter λ_1/λ_2 this divergence increases, at the limit (as $\lambda_1/\lambda_2 \to \infty$) reaching the maximum error of 28%.

NOTATION

$x = x'/\delta$ and $y = y'/\delta$	are dimensionless coordinates;
δ	is the thickness of the plate;
$\xi = a/2\delta$	is the dimensionless width of the inclusion;
τ	is the temperature of the plate or of the inclusion;
t	is the temperature of the ambient medium;
$\theta = (t_{in} - \tau)/(t_{in} - t_{out})$	is the dimensionless temperature;
α	is the heat-transfer coefficient;
λ_i	is the coefficient of thermal conductivity;
$Bi = \alpha_{in}\delta/\lambda_i$	is the Biot number;
$k = \alpha_{out}/\alpha_{in}$.	

Subscripts

i = 1	denotes the inclusion zone;
i = 2	denotes the plate zone;
in	denotes the inside (y = 0) surface;
out	denotes the outside y = 1) surface.

LITERATURE CITED

1. V. K. Ivashkova, Investigating the Thermal Engineering Properties of Barrier Structures by the Method of Electrical Simulation [in Russian], Gosstroiizdat, Moscow (1960).